A 12 PART WEBINAR SERIES ON:

COMPOSITE MATERIALS ENGINEERING

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- Ph.D. and M.A.Sc. in Composite Materials Engineering
- Over 15 years experience in industry and academia working on polymer matrix composites in aerospace, automotive, marine, energy, recreation and others
- Experience working with over 150 companies from SME to major international corporations
- Expertise in liquid composite moulding and thermal management

PAST WEBINAR RECORDINGS NOW POSTED

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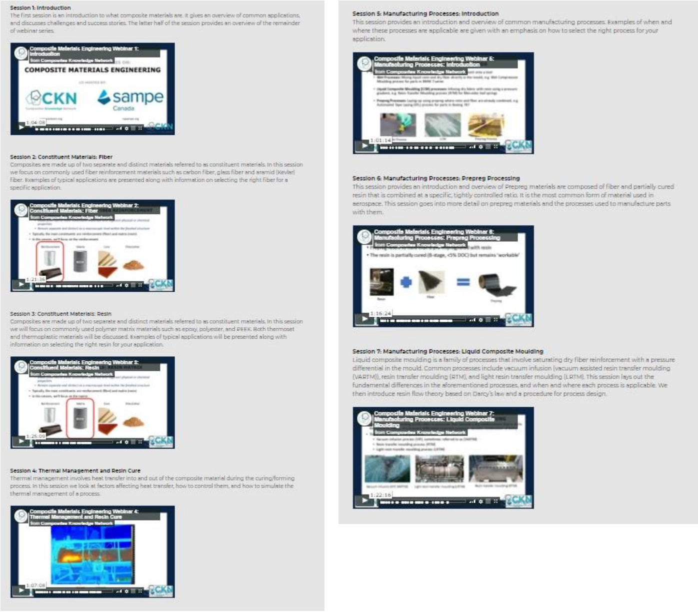

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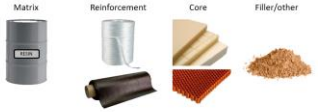
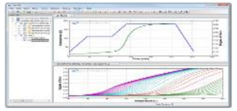
OVERVIEW OF WEBINAR SERIES

- **Series of 12 webinars, 1 hour each**

Introduction
Constituent Materials
Thermal Management

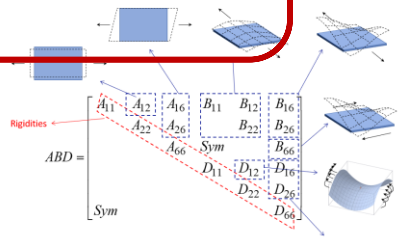




Processing (Manufacturing)
Prepreg Processing
Liquid Composite Moulding

Mechanics of Composites
Micromechanics
Lamina and Laminate Level
Failure

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ \kappa \end{Bmatrix}$$



Testing Composites
Common Defects

- **For more information on dates and times visit:**
  https://compositeskn.org/aimevents/

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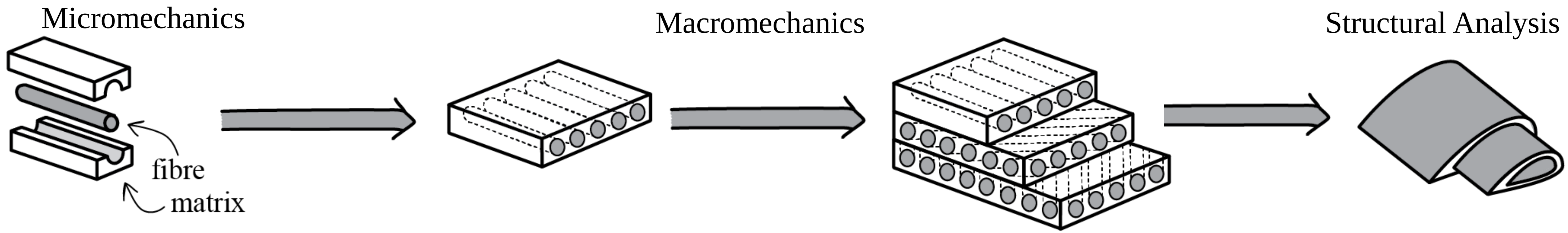
OUTCOMES OF THIS WEBINAR

- How can I predict my material properties based on my selection of resin and fibre?
- What effect does Vf have on material properties?
- What effect does fibre orientation have on material properties?

- **Upcoming webinars**
  - What effect does fibre, matrix, and stacking sequence have on material properties?
  - What effect does fibre, matrix, and stacking sequence have on strength?

ANALYSIS OF COMPOSITE MATERIALS

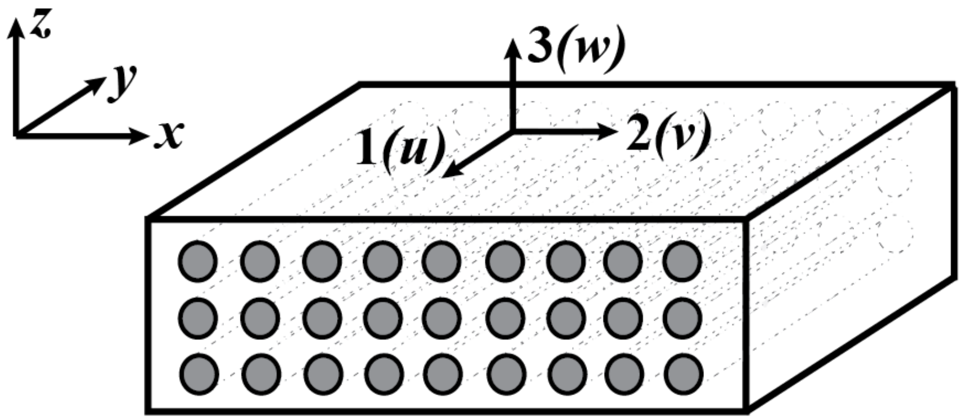
The complicated nature of a structure made from composite materials lends to many levels of analysis



- Micro-mechanics focuses on the stresses, strains, damage, etc., happening at the level of individual fibres and the fibre/matrix interface
- Macro-mechanics focuses on the analysis at the level of individual lamina and sub-laminates
- Structural mechanics involves analysis at the structural scale

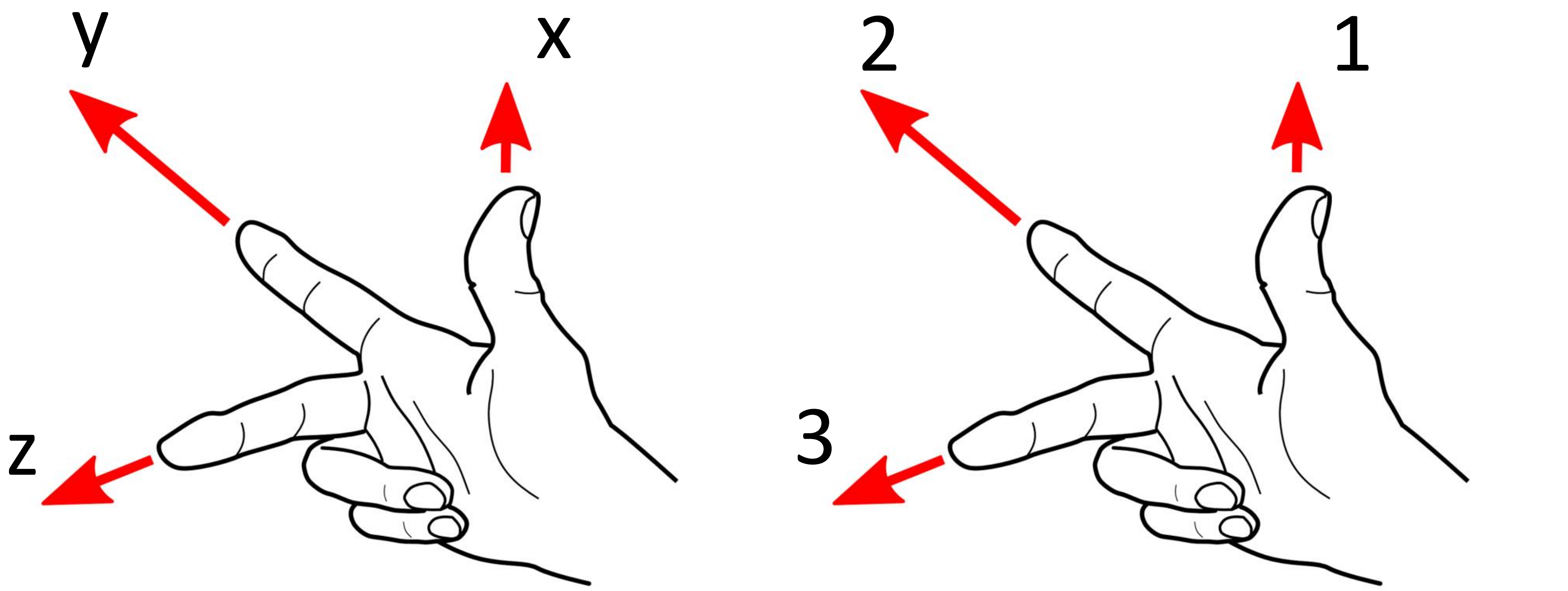
CONVENTIONS

- A common ‘dialect’ has been developed among those who perform design and analysis with composite materials
- The fibre direction in a lamina (or a single layer) is called the “1” direction
  - “u” = displacement in 1-direction
- The in-plane perpendicular to the fibre direction is called the “2” direction
  - “v” = displacement in 2-direction
- The out-of-plane perpendicular to the fibre direction is called the “3” direction
  - “w” = displacement in 3-direction



CONVENTIONS

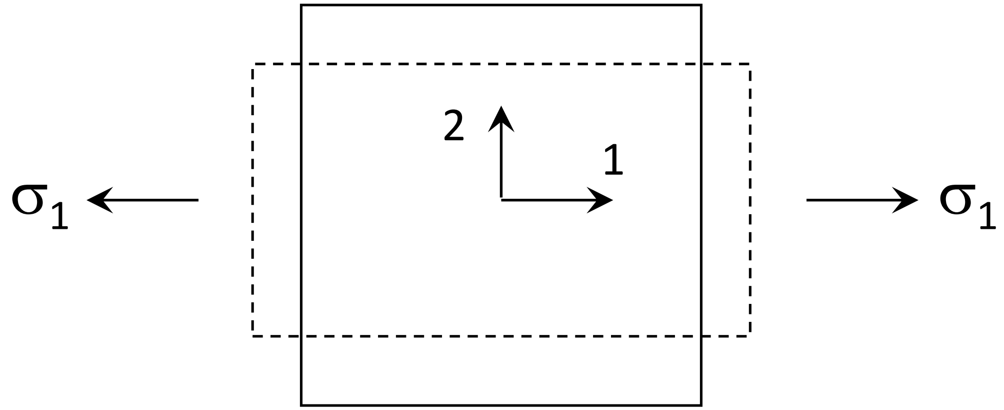
- Right-hand rule



HOOKE'S LAW: 1-DIRECTION

- Consider a linear elastic isotropic material (e.g. steel). It is subjected only to stresses in the 1- or 2- directions. This is known as *plane stress*. Plane stress is an assumption that is valid when the thickness of a body is small and there are no out-of-plane loads, typical conditions for a lamina.

- Now consider such a material subjected to an axial load:



HOOKE'S LAW: 1-DIRECTION

- The resulting strains can be written as:

$$\varepsilon_1 = \frac{\sigma_1}{E} \qquad \varepsilon_2 = -\nu\varepsilon_1 = -\nu\frac{\sigma_1}{E}$$

- Where *E* is the Young's modulus (modulus of elasticity) and ν is the Poisson's ratio. Recall that Poisson's ratio is defined as:

$$\nu = -\frac{\varepsilon_2}{\varepsilon_1}$$

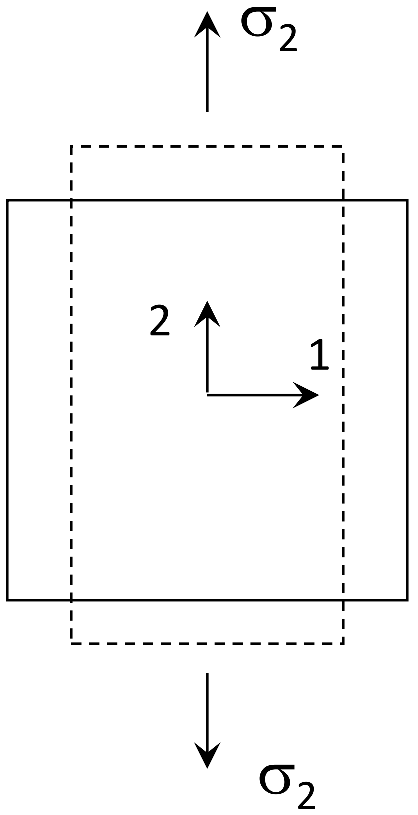
under uniaxial loading conditions.

HOOKE'S LAW: 2-DIRECTION

- Similarly, if a uniaxial load is considered in the 2-direction:

$$\varepsilon_1 = -\nu\varepsilon_2 = -\nu\frac{\sigma_2}{E}$$

$$\varepsilon_2 = \frac{\sigma_2}{E}$$

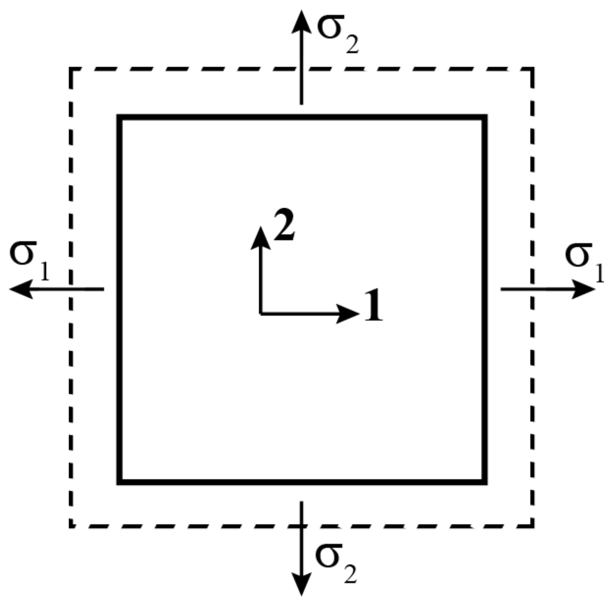


HOOKE'S LAW: BIAXIAL LOAD

- If both stresses are present, then we can express the strains by using the principle of superposition

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}$$

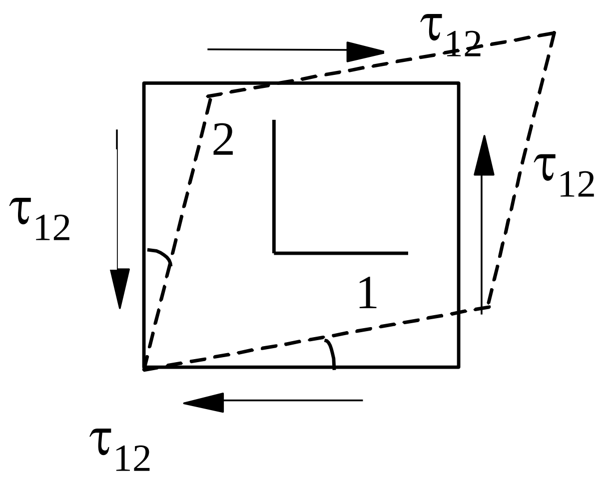
$$\varepsilon_2 = -\nu\frac{\sigma_1}{E} + \frac{\sigma_2}{E}$$



HOOKE'S LAW: SHEAR STRESS

- In a similar manner, consider the case where only a shear stress is applied

$$\gamma_{12} = \frac{\tau_{12}}{G}$$



- *G*, the shear modulus, can be expressed in terms of *E* and ν for isotropic materials (ie. only two independent elastic constants):

$$G = \frac{E}{2(1+\nu)}$$

HOOKE'S LAW

- The biaxial and pure shear loading cases can be combined and written in matrix form for an isotropic material:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \qquad \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

Compliance Matrix [S]

Stiffness Matrix [Q]

$$\{\varepsilon\} = [S]\{\sigma\}$$

$$\{\sigma\} = [Q]\{\varepsilon\}$$

$$[Q] = [S]^{-1}$$

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ORTHOTROPIC MATERIALS

- Composite materials are orthotropic, however, the same principles as isotropic apply.
- In the 1-direction, the strain equations are:

$$\varepsilon_1 = \frac{\sigma_1}{E_1} \qquad \varepsilon_2 = -\nu_{12}\varepsilon_1 = -\nu_{12}\frac{\sigma_1}{E_1}$$

- and, in the 2-direction:

$$\varepsilon_2 = \frac{\sigma_2}{E_2} \qquad \varepsilon_1 = -\nu_{21}\varepsilon_2 = -\nu_{21}\frac{\sigma_2}{E_2}$$

- Note that for unidirectional composites, $E_1 >> E_2$, and typically $\nu_{12} > \nu_{21}$. As a result, ν_{12} is the "major" Poisson's ratio, ν_{21} is the "minor" Poisson's ratio.

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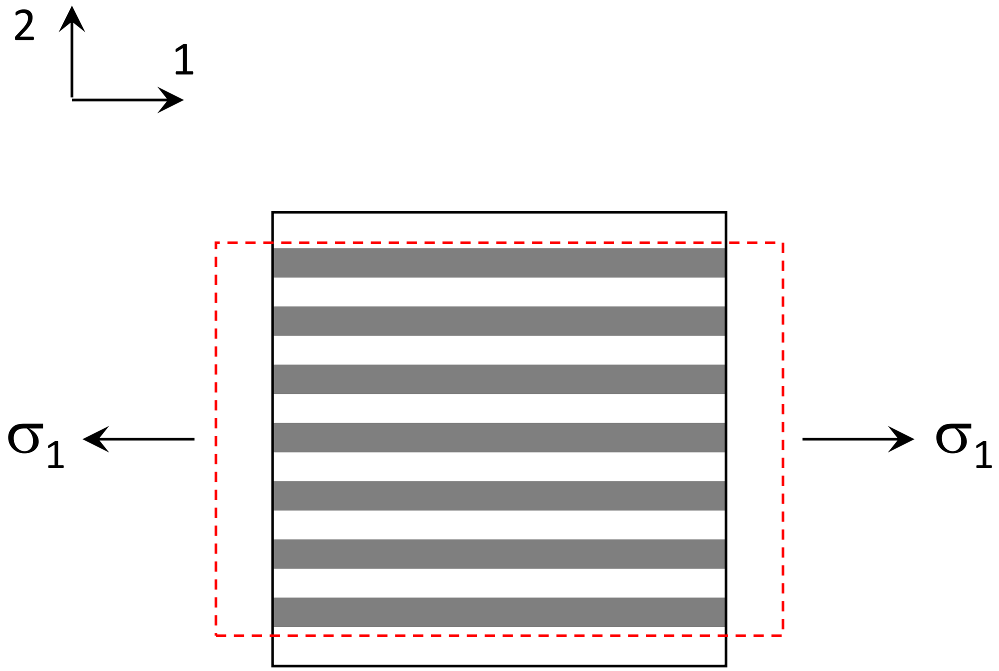
ORTHOTROPIC MATERIALS: POISSION'S RATIO



Effect of Major Poisson's Ratio, ν_{12}

Effect of Minor Poisson's Ratio, ν_{21}

HOOKE'S LAW FOR ORTHOTROPIC MATERIALS

- The stress-strain relationship can be written:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & \dfrac{-\nu_{21}}{E_2} & 0 \\ \dfrac{-\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

- However, for any elastic body, the compliance and stiffness matrices must be symmetric. In this case:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$

HOOKE'S LAW FOR ORTHOTROPIC MATERIALS

- Hooke's Law for a 2D unidirectional lamina is then:

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}=\underbrace{\begin{bmatrix}\frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & 0\\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}}\end{bmatrix}}_{\text{Compliance Matrix [S]}}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\underbrace{\begin{bmatrix}\frac{E_1}{1-\nu_{21}\nu_{12}} & \frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}} & 0\\ \frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}} & \frac{E_2}{1-\nu_{21}\nu_{12}} & 0\\ 0 & 0 & G_{12}\end{bmatrix}}_{\text{Stiffness Matrix [Q]}}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}$$

Compliance Matrix [S]

Stiffness Matrix [Q]

- Note that 4 independent constants are required:

E_1, E_2, (ν_{12} or ν_{21}), G_{12}

MICROMECHANICS

- Given a lamina, made from a specific matrix and a specific fibre, how are the engineering constants determined?

- The engineering constants depend on a number of variables:
  - The individual material constituents
  - The fibre volume fraction
  - The packing geometry
  - The processing method

MICROMECHANICS: VOLUME FRACTION

- $v_{c,f,m,v}$ = volume of composite, fiber, matrix, voids
- $\rho_{c,f,m}$ = density of composite, fiber, matrix
- The fiber volume fraction, V_f is:

$$V_f = \frac{v_f}{v_c}$$

- The matrix volume fraction, V_m is:

$$V_m = \frac{v_m}{v_c}$$

- The void volume fraction, V_v is:

$$V_v = \frac{v_v}{v_c}$$

- Note that:

$$V_f + V_m + V_v = 1$$

$$v_f + v_m + v_v = v_c$$

Note: V_* = volume fraction (big V)
v_* = volume of component (little v)

MICROMECHANICS: MASS FRACTION

- $w_{c,f,m}$ = mass of composite, fiber, matrix
- The fiber mass fraction, W_f is:

$$W_f = \frac{w_f}{w_c}$$

- The matrix mass fraction, w_m is:

$$W_m = \frac{w_m}{w_c}$$

- The total mass of the composite, w_c is:

$$w_c = w_f + w_m$$

Note: W_* = mass fraction (big W)
w_* = mass of component (little w)

MICROMECHANICS: DENSITY

- Knowing the volume and masses, we can determine the density:

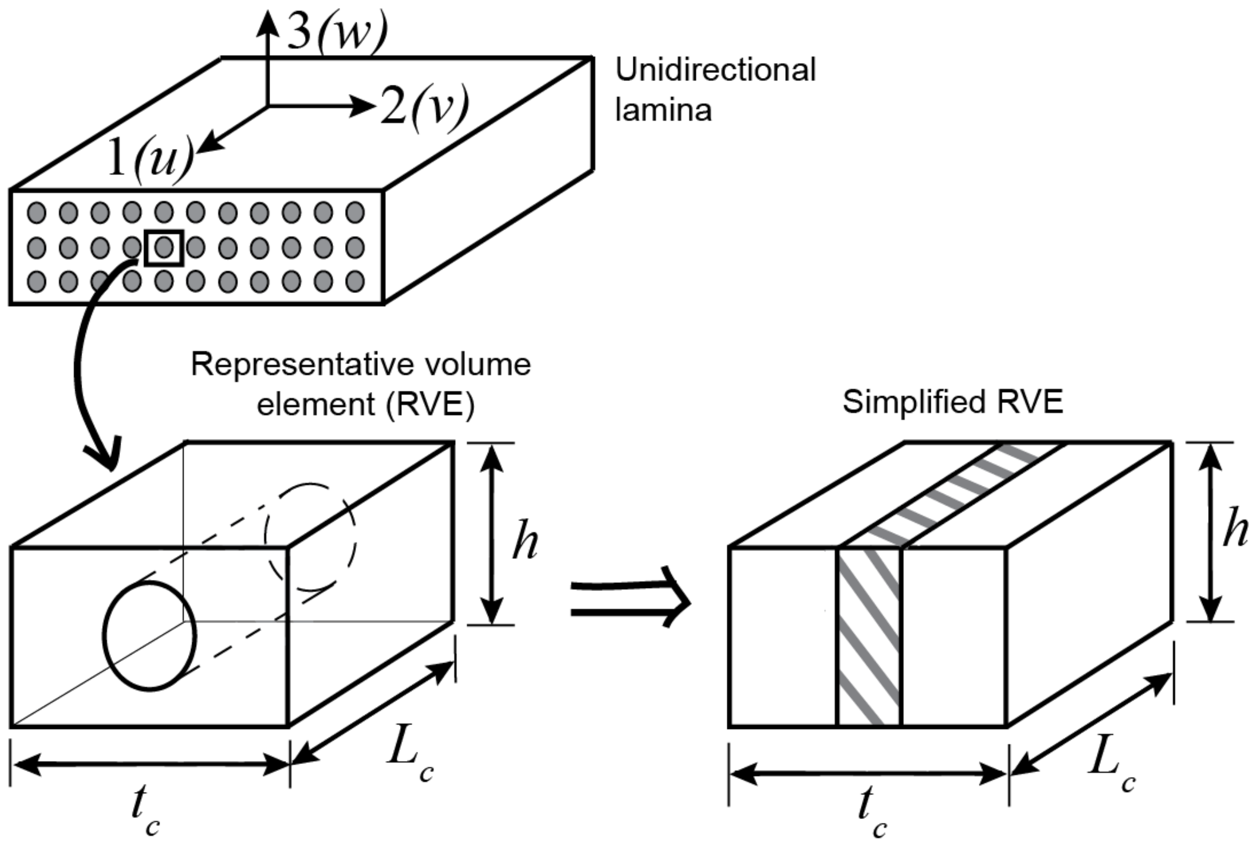
$$w_c = w_f + w_m$$
$$\rho_c v_c = \rho_f v_f + \rho_m v_m$$
$$\rho_c = \rho_f \frac{v_f}{v_c} + \rho_m \frac{v_m}{v_c}$$
$$\rho_c = \rho_f V_f + \rho_m V_m$$

Assuming voids are negligible:

$$\rho_c = \rho_f V_f + \rho_m (1 - V_f)$$

MICROMECHANICS: STRENGTH OF MATERIALS APPROACH

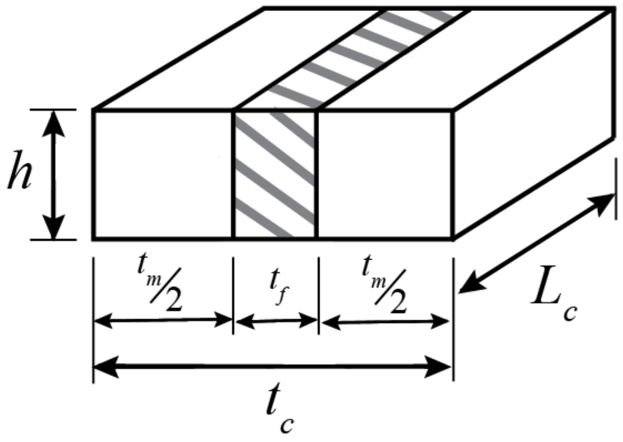
- Consider a representative volume element (RVE) taken from a unidirectional lamina:



MICROMECHANICS: SIMPLIFIED RVE REPRESENTATION

- The fibre, matrix and composite are assumed to be the same height, h

  but of thicknesses t_f, t_m, and t_c



- The area of each are:

$$A_f = t_f h \qquad A_m = t_m h \qquad A_c = t_c h$$

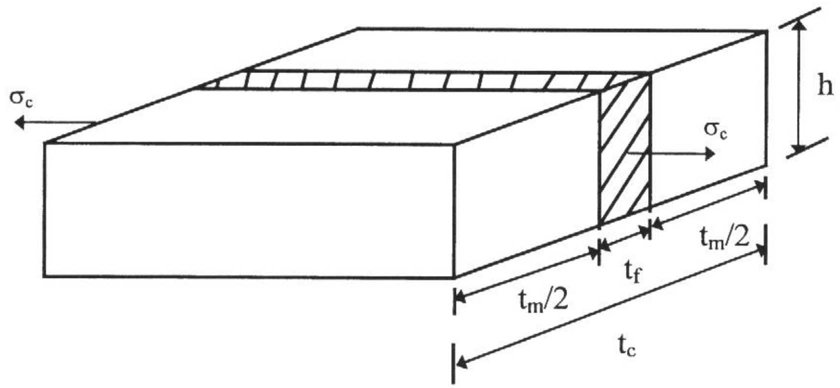
- The volume fractions are defined as:

$$V_f = \frac{A_f}{A_c} = \frac{t_f}{t_c} \qquad V_m = \frac{A_m}{A_c} = \frac{t_m}{t_c} = 1 - V_f$$

MICROMECHANICS: STRENGTH OF MATERIALS APPROACH

- The strength of materials approach (also known as the mechanics of materials approach) to micro-mechanics makes the following assumptions:
  - The bond between the fibres and matrix is perfect
  - The fibre diameters, and the space between the fibres are uniform
  - The fibres are continuous and parallel
  - The fibres and matrix are linear elastic, following Hooke’s Law
  - The fibres are all the same strength and the elastic moduli of the fibres and matrix are constant
  - The composite contains no voids

MICROMECHANICS: LONGITUDINAL YOUNG'S MODULUS



- Consider a unidirectional force applied to the simplified RVE in the fibre direction, such that it is shared by the fibre and the matrix

$$F_c = F_f + F_m$$

- These forces can also be expressed as stresses:

$$F_c = \sigma_c A_c \qquad F_f = \sigma_f A_f \qquad F_m = \sigma_m A_m$$

- With the assumption of linear-elastic, isotropic fibres and matrix we can write the stress-strain relationships as:

$$\sigma_c = E_1 \varepsilon_c \qquad \sigma_f = E_f \varepsilon_f \qquad \sigma_m = E_m \varepsilon_m$$

MICROMECHANICS: LONGITUDINAL YOUNG'S MODULUS

- Now, we can write:

$$F_c = F_f + F_m$$

$$E_1 \varepsilon_c A_c = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m$$

- But, our assumptions tell us:

$$\delta_c = \delta_f = \delta_m$$

- And since the lengths are the same:

$$\varepsilon_c = \varepsilon_f = \varepsilon_m$$

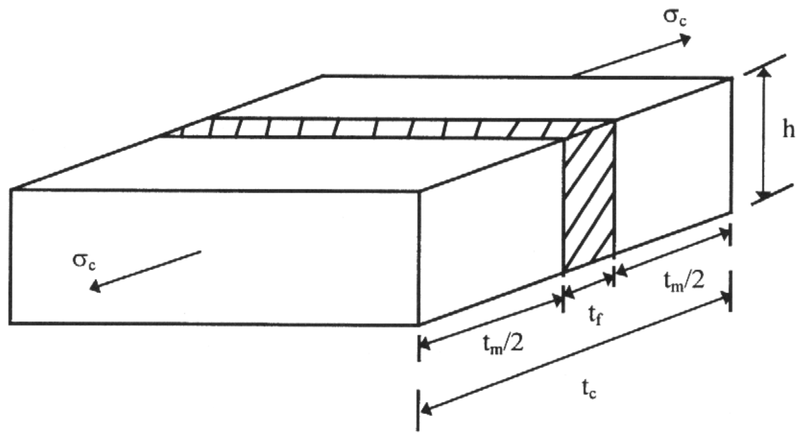
- So, we can write:

$$E_1 = E_f \frac{A_f}{A_c} + E_m \frac{A_m}{A_c}$$

- Or, recalling the volume fractions:

$$E_1 = E_f V_f + E_m V_m$$

MICROMECHANICS: TRANSVERSE YOUNG'S MODULUS



- Now consider a unidirectional force applied to the simplified RVE in the matrix direction

- In this case, the force in the matrix and the fibre is the same, but the displacements add up to the total displacement:

$$F_c = F_f = F_m \qquad \sigma_c = \sigma_f = \sigma_m$$

$$\delta_c = \delta_f + \delta_m$$

MICROMECHANICS: TRANSVERSE YOUNG'S MODULUS

- Unlike the longitudinal case, the thickness of the matrix and fibre are different, so the strains are not equal:

$$\delta_c = t_c \varepsilon_c \qquad \delta_f = t_f \varepsilon_f \qquad \delta_m = t_m \varepsilon_m$$

- But, we can express the strains according to Hooke's Law:

$$\varepsilon_c = \frac{\sigma_c}{E_2} \qquad \varepsilon_f = \frac{\sigma_f}{E_f} \qquad \varepsilon_m = \frac{\sigma_m}{E_m}$$

- Therefore, the displacement equation is:

$$t_c \frac{\sigma_c}{E_2} = t_f \frac{\sigma_f}{E_f} + t_m \frac{\sigma_m}{E_m}$$

MICROMECHANICS: TRANSVERSE YOUNG'S MODULUS

- However, the stresses are the same and:

$$\frac{t_f}{t_c} = \frac{t_f h}{t_c h} = \frac{A_f}{A_c} = V_f \qquad \frac{t_m}{t_c} = \frac{t_m h}{t_c h} = \frac{A_m}{A_c} = V_m$$

- Therefore:

$$t_c \frac{\sigma_c}{E_2} = t_f \frac{\sigma_f}{E_f} + t_m \frac{\sigma_m}{E_m}$$

$$\frac{1}{E_2} = \frac{1}{E_f} V_f + \frac{1}{E_m} V_m$$

- Rearranged:

$$E_2 = \frac{E_f E_m}{E_m V_f + E_f V_m}$$

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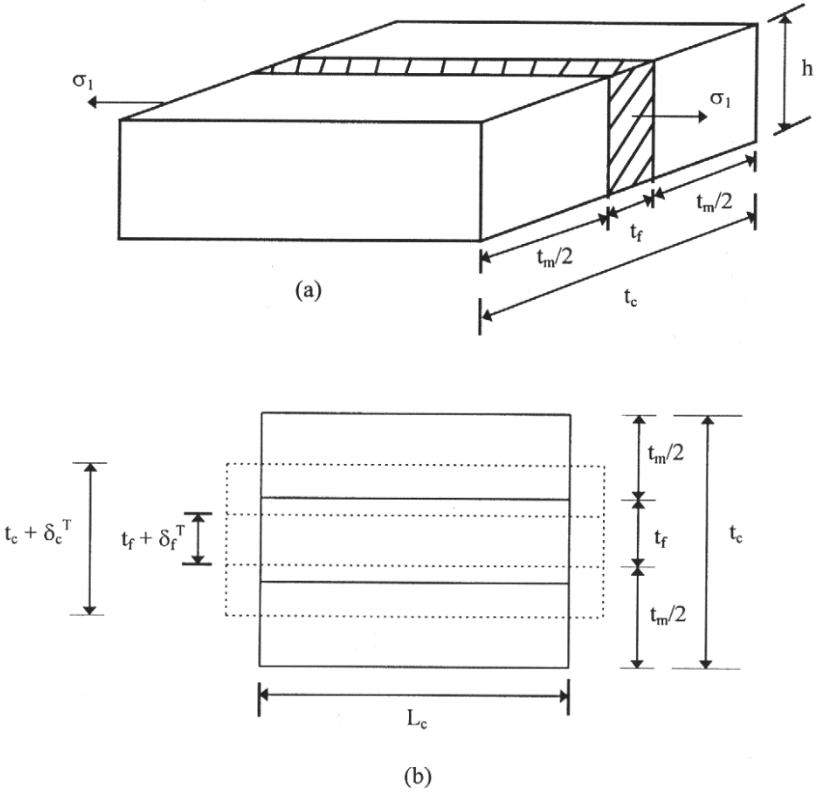
MICROMECHANICS: MAJOR POISSON'S RATIO



- Assume our RVE is loaded in the 1-direction (along the fibres)
- The *transverse* displacements are:

$$\delta_c^T = \delta_f^T + \delta_m^T$$

- The *transverse* strains are:

$$\varepsilon_c^T = \frac{\delta_c^T}{t_c} \quad \varepsilon_f^T = \frac{\delta_f^T}{t_f} \quad \varepsilon_m^T = \frac{\delta_m^T}{t_m}$$

- And, from the definition of Poisson's ratio:

$$\varepsilon_c^T = -\nu_{12}\varepsilon_c^L \qquad \nu_{LT} = \frac{-\varepsilon^T}{\varepsilon^L}$$

$$\varepsilon_f^T = -\nu_f \varepsilon_f^L$$

$$\varepsilon_m^T = -\nu_m \varepsilon_m^L$$

Note: Poisson's ratio is the Greek letter ν (nu), not the letter v

MICROMECHANICS: MAJOR POISSON'S RATIO

- So:

$$\delta_c^T = \delta_f^T + \delta_m^T$$

$$-t_c \nu_{12} \varepsilon_c^L = -t_f \nu_f \varepsilon_f^L - t_m \nu_m \varepsilon_m^L$$

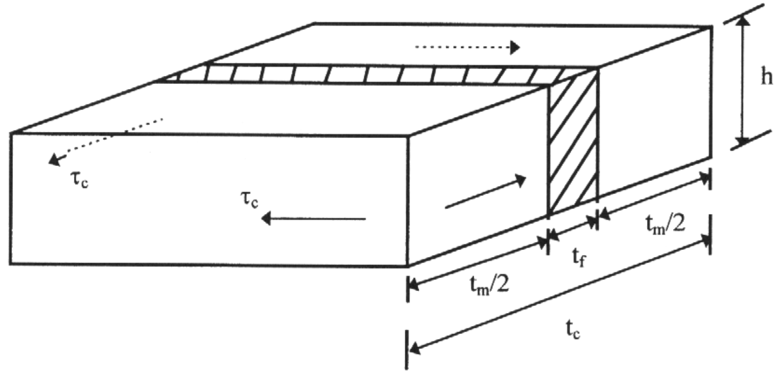
- But, when loading in the 1-direction, the strains are equal…

$$\nu_{12} = \nu_f \frac{t_f}{t_c} + \nu_m \frac{t_m}{t_c}$$

- And, recalling the volume fraction equations:

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

MICROMECHANICS: IN-PLANE SHEAR MODULUS



- Consider a pure shear stress, τ_c, applied to the RVE
- The resulting deformations are:

$$\delta_c = \delta_f + \delta_m$$

where the displacements are:

$$\delta_c = \gamma_c t_c \quad \delta_f = \gamma_f t_f \quad \delta_m = \gamma_m t_m$$

- Now, recalling Hooke's Law,

$$\gamma_c = \frac{\tau_c}{G_{12}} \qquad \gamma_f = \frac{\tau_f}{G_f} \qquad \gamma_m = \frac{\tau_m}{G_m}$$

therefore:

$$\frac{\tau_c}{G_{12}} t_c = \frac{\tau_f}{G_f} t_f + \frac{\tau_m}{G_m} t_m$$

$\Rightarrow$

$$\frac{1}{G_{12}} = \frac{1}{G_f}\frac{t_f}{t_c} + \frac{1}{G_m}\frac{t_m}{t_c}$$

$$\frac{1}{G_{12}} = \frac{1}{G_f} V_f + \frac{1}{G_m} V_m$$

MICROMECHANICS: SUMMARY

$$\rho_c = \rho_f V_f + \rho_m V_m$$

$$E_1 = E_f V_f + E_m V_m$$

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$

$$\frac{1}{E_2} = \frac{1}{E_f} V_f + \frac{1}{E_m} V_m$$

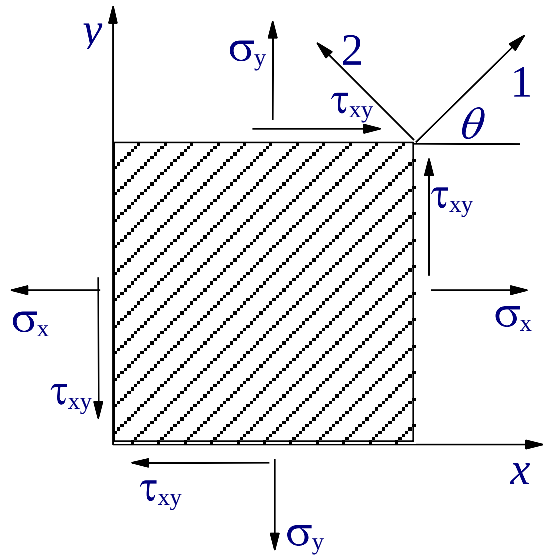
$$\frac{1}{G_{12}} = \frac{1}{G_f} V_f + \frac{1}{G_m} V_m$$

AKA "Rule of Mixtures"

$$X = X_1 V_1 + X_2 V_2 + \cdots + X_n V_n$$

ANGLE LAMINA

- When the material axes (1-2-3) of a lamina do not coincide with the loading axes (x-y-z), then the lamina is an "angle lamina"
- In order to determine the stress-strain relations in the global coordinate system (x-y-z), the elastic properties of the lamina have to be transformed from the local coordinate system (1-2-3)



(STRESS) TRANSFORMATION MATRIX

- Using a transformation matrix, stresses along the x-y directions can be related to the stresses in the 1-2 directions:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

where *s*=sinθ , and *c*=cosθ

- This can be written as:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

(STRAIN) TRANSFORMATION MATRIX

- Strains are transformed in the same manner:

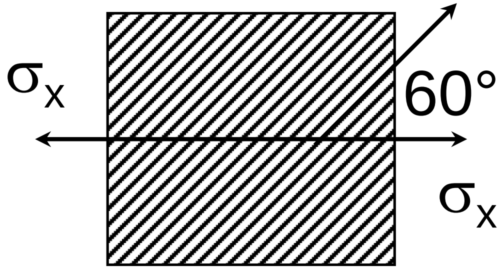
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \left[T^*\right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

- where [T*] is

$$\left[T^*\right] = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2 - s^2 \end{bmatrix}$$

STRESS/STRAIN TRANSFORMATION EXAMPLE

- A unidirectional lamina is oriented at 60°. A stress of 10 MPa is applied in the x-direction (0°)
  - Determine the stresses in the material direction (1-2)



$$[T]=\begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix}=\begin{bmatrix} \frac{1}{4} & \frac{3}{4} & \frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}=[T]\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}=[T]\begin{Bmatrix} 10 \\ 0 \\ 0 \end{Bmatrix}=\begin{Bmatrix} 2.5 \\ 7.5 \\ -4.33 \end{Bmatrix} \text{ MPa}$$

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STRESS/STRAIN TRANSFORMATION

- Conversion to the x-y-z coordinate system works in the same manner:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix}$$

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HOOKE'S LAW FOR ANGLE LAMINA

- Recall Hooke's Law:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

- We need to establish a similar relationship:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\bar{S}] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Note: $[\bar{S}]$ is referred to as the transformed compliance matrix

- First, replace the stress and strain terms:

$$[T^*] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [S][T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

HOOKE'S LAW FOR ANGLE LAMINA

- Now, pre-multiply by the inverse of [T*]

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \left[T^*\right]^{-1}[S][T]\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

- So,

$$\left[\bar{S}\right] = \left[T^*\right]^{-1}[S][T]$$

- Now, [S] is a compliance matrix in the x-y-z system:

$$\left[\bar{S}\right] = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ & \bar{S}_{22} & \bar{S}_{26} \\ Sym & & \bar{S}_{66} \end{bmatrix}$$

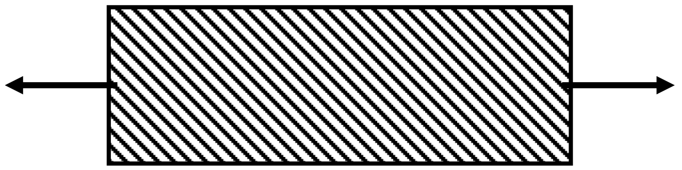
Note: $\left[\bar{S}\right]$ is referred to as the transformed compliance matrix

COMPLIANCE MATRIX FOR ANGLE LAMINA

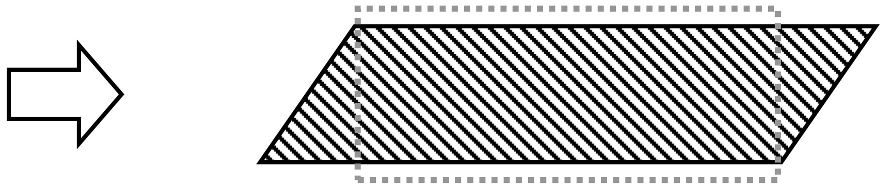
$$[\bar{S}] = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ & \bar{S}_{22} & \bar{S}_{26} \\ Sym & & \bar{S}_{66} \end{bmatrix}$$

Non-zero terms in an angle lamina means that there is coupling between shear and axial directions

What does this mean in reality?



Applied normal stress



Coupled deformation (extension and shear)

ENGINEERING CONSTANTS FOR ANGLE LAMINA

- Based on the compliance matrix of an angle lamina, engineering constants along x-y directions (E_x, E_y, v_{xy}, G_{xy}) are defined as follows:

$$\left[\bar{S}\right]=\begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ & \bar{S}_{22} & \bar{S}_{26} \\ Sym & & \bar{S}_{66} \end{bmatrix}=\begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{m_x}{E_1} \\ & \frac{1}{E_y} & -\frac{m_y}{E_1} \\ Sym & & \frac{1}{G_{xy}} \end{bmatrix}$$

- m_x and m_y are shear coupling parameters

ENGINEERING CONSTANTS FOR ANGLE LAMINA

$$[\bar{S}] = [T^*]^{-1}[S][T] \Rightarrow$$

$$\frac{1}{E_x} = \bar{S}_{11} = \frac{1}{E_1}c^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)s^2c^2 + \frac{1}{E_2}s^4$$

$$\frac{1}{E_y} = \bar{S}_{22} = \frac{1}{E_1}s^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)s^2c^2 + \frac{1}{E_2}c^4$$

$$\frac{1}{G_{xy}} = \bar{S}_{66} = 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^2c^2 + \frac{1}{G_{12}}(s^4 + c^4)$$

$$\nu_{xy} = -E_x\bar{S}_{12} = E_x\left[\frac{\nu_{12}}{E_1}(s^4 + c^4) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)s^2c^2\right]$$

$$m_x = -\bar{S}_{16}E_1 = -E_1\left[\left(\frac{-2}{E_1} - \frac{-2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)sc^3 - \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^3c\right]$$

$$m_y = -\bar{S}_{26}E_1 = -E_1\left[\left(\frac{-2}{E_1} - \frac{-2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^3c - \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)sc^3\right]$$

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HOOKE'S LAW FOR ANGLE LAMINA

- Up to now, we've expressed everything in terms of compliance, but we can express them in terms of stiffness too:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$[T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [Q][T^*] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][T^*] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\overline{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ & \overline{Q}_{22} & \overline{Q}_{26} \\ Sym & & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Note: $[\overline{Q}]$ is referred to as the transformed stiffness matrix

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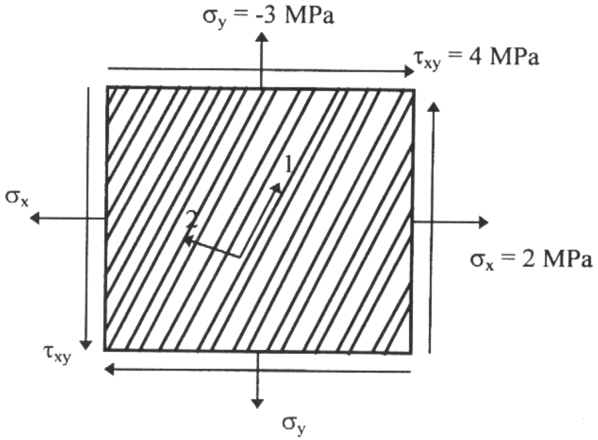
ANGLE LAMINA EXAMPLE

- A 60° angle lamina is made from carbon/epoxy with properties as shown below
- With the applied stresses, determine:
  - the global strains
  - the local strains
  - the local stresses
  - the equivalent global engineering constants



V_f	0.70
E_1	181 GPa
E_2	10.30 GPa
ν_{12}	0.28
G_{12}	7.17 GPa

ANGLE LAMINA EXAMPLE SOLUTION

- First, find cos q , sin q

$$c = \cos(60°) = 0.500$$
$$s = \sin(60°) = 0.866$$

- Now, find the local compliance matrix

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

ANGLE LAMINA EXAMPLE SOLUTION

$$S_{11} = \frac{1}{181} = 5.52 \cdot 10^{-3}\ \text{GPa}^{-1}$$

$$S_{12} = -\frac{0.28}{181} = -1.55 \cdot 10^{-3}\ \text{GPa}^{-1}$$

$$S_{22} = \frac{1}{10.3} = 97.1 \cdot 10^{-3}\ \text{GPa}^{-1}$$

$$S_{66} = \frac{1}{7.17} = 139 \cdot 10^{-3}\ \text{GPa}^{-1}$$

$$[S] = \begin{bmatrix} 5.52 & -1.55 & 0 \\ & 97.1 & 0 \\ Sym & & 139 \end{bmatrix} \cdot 10^{-3}\ \text{GPa}^{-1}$$

And, the local stiffness matrix…

$$[Q] = [S]^{-1} = \begin{bmatrix} 181.8 & 2.897 & 0 \\ & 10.35 & 0 \\ Sym & & 7.17 \end{bmatrix} \text{GPa}$$

ANGLE LAMINA EXAMPLE SOLUTION

$$[S]=\begin{bmatrix} 5.52 & -1.55 & 0 \\ & 97.1 & 0 \\ Sym & & 139 \end{bmatrix}\cdot 10^{-3}\ \text{GPa}^{-1} \qquad [\bar{S}]=[T^*]^{-1}[S][T]$$

The global compliance matrix requires the transformation matrices…

$$[T]=\begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix}=\begin{bmatrix} 0.25 & 0.75 & 0.866 \\ 0.75 & 0.25 & -0.866 \\ -0.433 & 0.433 & -0.50 \end{bmatrix}$$

$$[T^*]=\begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2-s^2 \end{bmatrix}=\begin{bmatrix} 0.25 & 0.75 & 0.433 \\ 0.75 & 0.25 & -0.433 \\ -0.866 & 0.866 & -0.50 \end{bmatrix}$$

ANGLE LAMINA EXAMPLE SOLUTION

And, the global compliance matrix becomes…

$$[\bar{S}] = [T^*]^{-1}[S][T] = \begin{bmatrix} 80.53 & -7.878 & -32.34 \\ & 34.75 & -46.96 \\ Sym & & 114.1 \end{bmatrix} \cdot 10^{-3}\ \text{GPa}^{-1}$$

The global stiffness matrix is the inverse…

$$[\bar{Q}] = [\bar{S}]^{-1} = \begin{bmatrix} 23.65 & 32.46 & 20.05 \\ & 109.4 & 54.19 \\ Sym & & 36.74 \end{bmatrix} \text{GPa}$$

ANGLE LAMINA EXAMPLE SOLUTION

The engineering constants are found from the global compliance matrix…

$$\frac{1}{\bar{S}_{11}} = E_x = 12.4\,\text{GPa} \qquad \nu_{xy} = -E_x \bar{S}_{12} = 0.0977$$

$$\frac{1}{\bar{S}_{22}} = E_y = 28.8\,\text{GPa} \qquad m_x = -\bar{S}_{16} E_1 = 5.85$$

$$\frac{1}{\bar{S}_{66}} = G_{xy} = 8.76\,\text{GPa} \qquad m_y = -\bar{S}_{26} E_1 = 8.50$$

The global strains are easily computed:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \left[\bar{S}\right] \begin{Bmatrix} 2 \\ -3 \\ 4 \end{Bmatrix} \text{MPa} = \begin{Bmatrix} 55.34 \\ -307.8 \\ 532.8 \end{Bmatrix} \mu\varepsilon$$

ANGLE LAMINA EXAMPLE SOLUTION

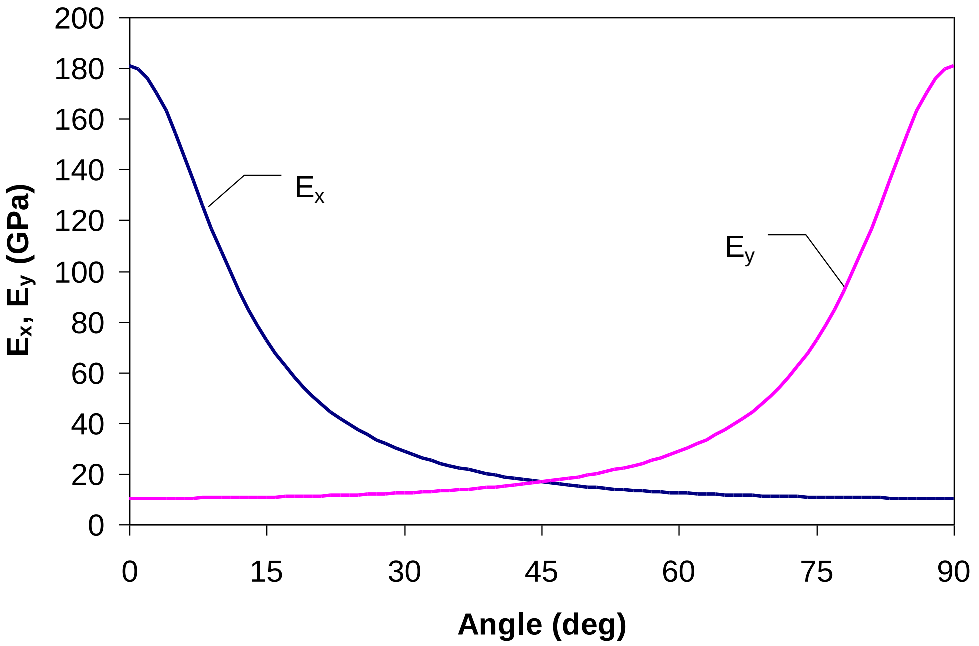
The local strains can be found by transforming the global strains:

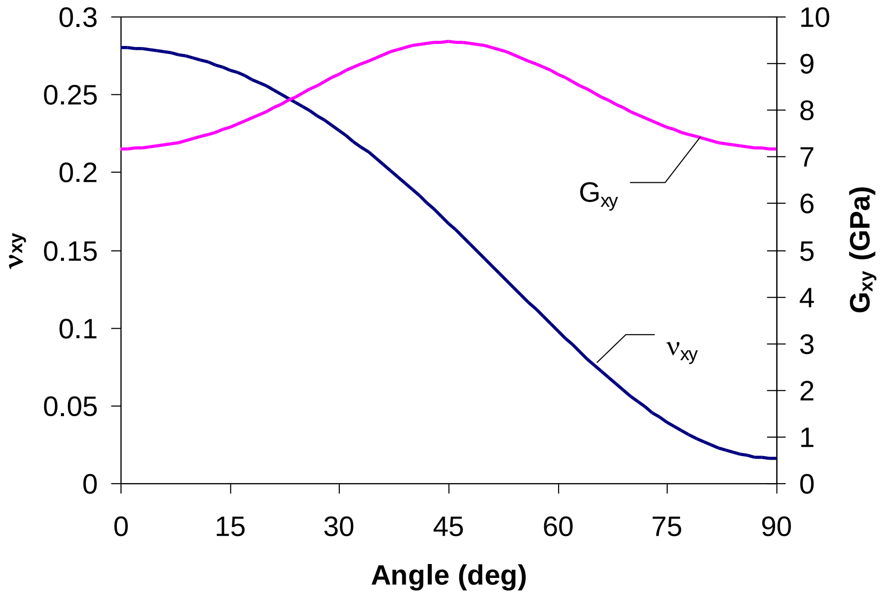
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \left[T^*\right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 13.67 \\ -266.2 \\ -580.9 \end{Bmatrix} \mu\varepsilon$$

The local stresses could be found by transforming the global stresses or by multiplying the local strains by the local stiffness matrix:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 1.714 \\ -2.714 \\ -4.165 \end{Bmatrix} \text{MPa}$$

ENGINEERING CONSTANTS AS A FUNCTION OF ANGLE





Thank you for joining us!

The next session is:

Session 8: Mechanics of Composites Part 2: Laminate Level

September 16, 2020 @ 9:00 am PT

Questions?

For more information on future dates and times visit:

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